

Combinatorial formula for Macdonald polynomials, Bethe Ansatz, and generic Macdonald polynomials

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Abstract

We give a direct proof of the combinatorial formula for interpolation Macdonald polynomials by introducing certain polynomials, which we call generic Macdonald polynomials, which depend on d additional parameters and specialize to all Macdonald polynomials of degree d . The form of these generic polynomials is that of a Bethe eigenfunction and they imitate, on a more elementary level, the R -matrix construction of quantum immanants.

1 Introduction

1.1 Interpolation Macdonald polynomials and combinatorial formula

1.1.1 The polynomials I_μ

Consider the symmetric polynomials

$$I_\mu(x_1, \dots, x_n; q, t, s) \in \mathbb{Q}(q, t, s)[x_1, \dots, x_n]^{S(n)},$$

where μ is a partition with at most n parts, satisfying the following interpolation condition: for any partition λ such that $\mu \not\leq \lambda$

$$I_\mu(\boldsymbol{\lambda}) = 0, \quad \boldsymbol{\lambda} = ([1, \lambda_1], \dots, [n, \lambda_n]), \quad (1.1)$$

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where

$$[i, j] = \frac{sq^{j+1}}{t^{i+1}} + \frac{t^{i+1}}{sq^{j+1}}.$$

Together with the condition that the degree of I_μ is at most $|\mu|$ this determines I_μ uniquely up to a constant factor. These polynomials I_μ were introduced and studied in [10]; they specialize to interpolation polynomials studied by Knop, Olshanski, Sahi, and the author in a series of papers, see the References. We will call the polynomials I_μ the interpolation Macdonald polynomials.

1.1.2 Combinatorial formula

The polynomials I_μ are very distinguished multivariate special functions with a large number of deep properties and important applications (in particular, to the better known orthogonal Macdonald polynomials). Most of their properties can be quite easily deduced from the following explicit formula for these polynomials which is known as the combinatorial formula.

Consider the diagram of the partition μ and a function

$$f : \square \mapsto \{1, \dots, n\}$$

from the squares of this diagram to integers. The function f is called a *reverse tableau* if its values weakly decrease along the rows and strictly decrease along the columns of μ . We have the following formula [10]

$$I_\mu = \sum_{\text{reverse tableaux } f} \Psi_f \prod_{\square \in \mu} \left(x_{f(\square)} - \frac{sq^{\text{col}(\square)}}{t^{f(\square)+\text{row}(\square)}} - \frac{t^{f(\square)+\text{row}(\square)}}{sq^{\text{col}(\square)}} \right), \quad (1.2)$$

where $\text{col}(\square)$ and $\text{row}(\square)$ denote the the column and row numbers a square $\square \in \mu$ and Ψ_f is a certain weight of the tableau f which will be specified below. It is the same weight as appears in the combinatorial formula for the ordinary Macdonald polynomials, which makes it clear that the ordinary Macdonald polynomial is the top degree term of I_μ .

1.1.3 Direct proof of the combinatorial formula

One of the main results of this note is a proof of the combinatorial formula (1.2) which does not use any properties of I_μ except for their definition. We recall that a great deal of nontrivial properties of I_μ were used in [10] in the

proof of (1.2). Now many of these properties can now be without difficulty deduced from (1.2) without falling into a vicious circle.

It is obvious that the right-hand side of (1.2) is a polynomial of degree $|\mu|$ and it is also easy to check that it satisfies the vanishing condition (1.1), see for example Lemma 4.1 in [11]. What remains is the nontrivial task of checking that this polynomial is symmetric in x_1, \dots, x_n .

1.1.4 How we prove the symmetry

Our way of verifying the symmetry in (1.2) will be the following. We introduce certain polynomials

$$I_U(x) \in \mathbb{Q}(u_1, \dots, u_d, q, t, s)[x_1, \dots, x_n]$$

of degree d in x . These polynomials have the property that for any μ such that $|\mu| = d$ one can tune the parameters $\{u_i\}$ so that to make I_U go into the right-hand side of (1.2). Because of this property, we call the polynomials I_U the *generic Macdonald polynomials*.

The additional freedom which comes with the parameters $\{u_i\}$ makes it easy to check that the polynomials $I_U(x)$ are symmetric. This is done in Section 2. After that, in Section 3 we show how to specialize the polynomials I_U to polynomials in the right-hand side of (1.2) which, in particular, completes our new proof of (1.2).

1.2 Generic Macdonald polynomials

1.2.1 Shifted Schur functions and Quantum Immanants

The polynomials I_U are not only a convenient technical tool but also an interesting object of study in their own right.

In order to better motivate their definition, we begin with the special case when $q = t \rightarrow 1$ and $s \rightarrow \infty$. In this limit, the polynomials I_μ become (after a change of variables) the shifted Schur functions s_μ^* , see [12], for which the combinatorial formula is

$$s_\mu^*(x) = \sum_{\text{reverse tableaux } f} \prod_{\square \in \mu} (x_{f(\square)} - c(\square)) , \quad (1.3)$$

where

$$c(\square) = \text{col}(\square) - \text{row}(\square)$$

denotes the *content* of a square $\square \in \mu$.

The functions s_μ^* have a very nice and important representation-theoretic interpretations as the image under the Harish-Chandra homomorphism of a distinguished linear basis of the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_n)$. The corresponding central elements are known as the *quantum immanants* and denoted by \mathbb{S}_μ .

Explicit formulas for quantum immanants were obtained in [6], see also [5, 7]. These formulas can be viewed as a noncommutative analog, or quantization, of the combinatorial formula (1.3), see for example Section 3.7 of [6]. We recall from [5] the following construction of \mathbb{S}_μ which was motivated by representation theory of the Yangian $Y(\mathfrak{gl}_n)$.

Consider the algebra of matrices with entries in $\mathcal{U}(\mathfrak{gl}_n)$. Let $E = (E_{ij})$ be the matrix formed by the standard basis elements of \mathfrak{gl}_n . Denote $|\mu|$ by d and consider the following expression

$$\mathrm{tr} (E - u_1) \otimes (E - u_2) \otimes \cdots \otimes (E - u_d) \cdot \overrightarrow{\prod}_{i,j} R_{ij}(u_i - u_j), \quad (1.4)$$

where

$$R_{ij}(u) = 1 - \frac{(ij)}{u} \in \mathbb{Q}(u) S(d)$$

is the rational R -matrix, the ordered product in (1.4) ranges over all pairs $1 \leq i < j \leq d$ ordered lexicographically, the symmetric group $S(d)$ acts on the d -fold tensor product by the usual permutation of indices, and the trace means, as usual, the sum of the diagonal matrix elements.

The expression (1.4) is an element of

$$\mathcal{U}(\mathfrak{gl}_n) \otimes \mathbb{Q}(u_1, \dots, u_d).$$

It is explained in [5] how (1.4) turns into \mathbb{S}_μ when the parameters $\{u_i\}$ approach, in a certain special way, the contents of the diagram μ . In this limit, the product of R -matrices in (1.4) turns into a projection onto a certain specific vector in the irreducible representation of $S(d)$ corresponding to μ .

1.2.2 Formula for generic Macdonald polynomials

The definition of the polynomials I_U is our attempt to keep as much as possible from (1.4) in the case of general q and t . This general case has two main features. On the one hand, in the absence of a simple structure to

replace $\mathcal{U}(\mathfrak{gl}_n)$ for the general q and t , we do not go over the Harish-Chandra bridge and stay entirely within the commutative world of polynomials. On the other hand, the additional flexibility provided by parameters q and t allows us to work with polynomial in a way very similar to what was done in the noncommutative world of $\mathcal{U}(\mathfrak{gl}_n)$.

As a commutative replacement for the R -matrix $R(u)$, we introduce the following function

$$\rho(u) = \frac{(1 - qu)(1 - tu/q)}{(1 - u)(1 - tu)}$$

and define the polynomial I_U to be the following analog of (1.4)

$$I_U(x_1, \dots, x_n) = \sum_{i_1, \dots, i_d=1}^n \prod_{k=1}^d \left(x_{i_k} + \frac{u_k}{t^{i_k}} + \frac{t^{i_k}}{u_k} \right) \prod_{i_k < i_l} \rho(u_k/u_l), \quad (1.5)$$

where the summation replaces the trace, first product replaces the tensor product, and the second product replaces the product of R -matrices.

Of course, the general structure of the formula (1.5) is a typical form of a Bethe eigenfunction.

1.2.3 Properties of generic Macdonald polynomials

As already mentioned, in Theorem 1 below we prove that polynomials I_U are symmetric in x_1, \dots, x_n . The proof of the symmetry is elementary and based on the study of the singularities of I_U in parameters $\{u_i\}$.

In Section 3 we show that as the parameters $\{u_i\}$ approach the (q, t) -analogs of the contents of a diagram μ , that is, as

$$\{u_i\} \rightarrow \left\{ \text{const } q^{\text{col}(\square)} t^{-\text{row}(\square)} \right\}_{\square \in \mu} \quad (1.6)$$

the generic Macdonald polynomial becomes the interpolation Macdonald polynomial labeled by μ , see Theorem 2. The limit (1.6) has to be taken in a certain special way which is parallel to what one does to obtain S_μ as a limit of (1.4).

Finally, we mention an important but potentially difficult open problem which is to expand the generic polynomial I_U in terms of the polynomials I_μ .

2 The polynomials $I_U(X)$

2.1 Definition

Let $X = \{x_1, \dots, x_n\}$ and $U = \{u_1, \dots, u_d\}$ be two finite sets of variables. Define a polynomial $I_U(x)$ of degree $d = |U|$ in the variables X with coefficients in rational functions in U as follows.

Introduce the following function

$$\rho(v) = \frac{(1 - qv)(1 - tv/q)}{(1 - v)(1 - tv)},$$

where q and t are parameters. Given a function f

$$f : \{1, \dots, d\} \rightarrow \{1, \dots, n\}$$

set, by definition,

$$R_f = \prod_{f(i) < f(j)} \rho(u_i/u_j),$$

where the product is over all pairs i, j in $\{1, \dots, d\}$ such that $f(i) < f(j)$. The polynomial $I_U(x)$ is defined by the following formula

$$I_U(X) = \sum_f R_f \prod_{i=1}^d \left(x_{f(i)} + \frac{u_i}{t^{f(i)}} + \frac{t^{f(i)}}{u_i} \right), \quad (2.1)$$

where the summation is over all n^d possible functions f . It is clear that the definition (2.1) is just another way of writing the formula (1.5) from the Introduction. We call these polynomials I_U the *generic Macdonald polynomials*.

From definition of $I_U(X)$, the following decomposition property of this polynomial is obvious. Fix any $k = 1, \dots, n$ and set $X' = \{x_1, \dots, x_k\}$. Then

$$I_U(X) = \sum_{U' \subset U} I_{U'}(X') I_{U \setminus U'}(X \setminus X') \prod_{u_i \in U', u_j \in U \setminus U'} \rho(u_i/u_j), \quad (2.2)$$

where the summation is over all subsets $U' \subset U$.

2.2 Symmetry

Since no ordering on U was used, the polynomial $I_U(X)$ is obviously invariant under permutations of the u_i 's. Our main result about the polynomials $I_U(X)$ is that it is also invariant under permutations of the x_i 's.

Theorem 1. *The polynomial $I_U(X)$ is symmetric in x_1, \dots, x_n .*

In the proof, we shall need some elementary properties of the function $\rho(v)$ which we collect in the following

Lemma 1. *We have*

$$\rho(v) = \rho\left(\frac{1}{tv}\right), \quad (2.3)$$

$$\rho(v) = 1 + \frac{(q-1)(q-t)}{q}v + O(v^2), \quad v \rightarrow 0, \quad (2.4)$$

$$\rho(v) = 1 + \frac{(q-1)(q-t)}{qt} \frac{1}{v} + O\left(\frac{1}{v^2}\right), \quad v \rightarrow \infty. \quad (2.5)$$

Proof of Theorem 1. It suffices to check that $I_U(X)$ is invariant under a transposition of two adjacent x_i 's. By the decomposition property (2.2), this reduces to showing that $I_U(x_1, x_2)$ is symmetric in x_1 and x_2 , identically in u_1, \dots, u_d for any d .

Let us single out one of the u_i 's and write

$$g(u_1) = I_U(x_1, x_2) - I_U(x_2, x_1),$$

where x_1, x_2 , and the rest of the u_i 's are considered as parameters. Our goal is to show that $g = 0$ identically.

The function g is a rational function of u_1 with at most first order poles at the following points

$$u_1 \in \{0, \infty, u_i, t^{\pm 1}u_i\}, \quad i = 2, \dots, d.$$

First we show that, in fact, this function is regular everywhere. By symmetry, it suffices to consider the points $\{0, \infty, u_2, t^{\pm 1}u_2\}$.

Observe that on the divisor $u_1 = u_2$ the function $I_U(x_1, x_2)$ itself is regular. Indeed, a function which is symmetric in u_1 and u_2 cannot have a pole of exact order one on the diagonal $u_1 = u_2$.

Next, consider the divisor $u_1 = t^{-1}u_2$. Only those functions f which satisfy

$$f(1) = 1, \quad f(2) = 2$$

contribute to the pole of $I_U(x_1, x_2)$ at $u_1 = t^{-1}u_2$. For such function f and any $i = 3, \dots, d$ we have either $f(i) = f(1) < f(2)$ or $f(1) < f(2) = f(i)$. The condition $u_1 = t^{-1}u_2$ and (2.3) yield that

$$\rho\left(\frac{u_i}{u_2}\right) = \rho\left(\frac{u_1}{u_i}\right).$$

Therefore

$$R_f = R_{f|_{\{3, \dots, d\}}} \prod_{i=2}^d \rho\left(\frac{u_1}{u_i}\right),$$

where $f|_{\{3, \dots, d\}}$ denotes the restriction of f to $\{3, \dots, d\}$. It follows that the residue of $I_U(x_1, x_2)$ at $u_1 = t^{-1}u_2$ is proportional to

$$\left(x_1 + \frac{u_1}{t} + \frac{t}{u_1}\right) \left(x_2 + \frac{u_2}{t^2} + \frac{t^2}{u_2}\right) I_{\{u_3, \dots, u_d\}}(x_1, x_2),$$

which by the condition $u_1 = t^{-1}u_2$ and induction on the number of u_i 's is symmetric in x_1 and x_2 .

In other words, $g(u_1)$ is regular at $u_1 = t^{-1}u_2$. By the symmetry between u_1 and u_2 , it is also regular at $u_1 = tu_2$.

Now consider the point $u_1 = 0$. Since for any function f we either have $f(1) = 1$ or $f(1) = 2$, it follows from the definition of $I_U(x_1, x_2)$ that

$$I_U(x_1, x_2) = \sum_{f'} R_{f'} \Pi_{f'} \left[\left(x_1 + \frac{u_1}{t} + \frac{t}{u_1}\right) \prod_{f'(i)=2} \rho\left(\frac{u_1}{u_i}\right) + \left(x_2 + \frac{u_1}{t^2} + \frac{t^2}{u_1}\right) \prod_{f'(i)=1} \rho\left(\frac{u_i}{u_1}\right) \right], \quad (2.6)$$

where the summation is over all possible function

$$f' : \{2, \dots, d\} \rightarrow \{1, 2\},$$

and where we use the following abbreviation

$$\Pi_{f'} = \prod_{i=2}^d \left(x_{f'(i)} + \frac{u_i}{t^{f'(i)}} + \frac{t^{f'(i)}}{u_i} \right).$$

From (2.4) we have

$$\begin{aligned} \left(x_1 + \frac{u_1}{t} + \frac{t}{u_1}\right) \prod_{f'(i)=2} \rho\left(\frac{u_i}{u_1}\right) = \\ \frac{t}{u_1} + x_1 + \frac{t(q-1)(q-t)}{q} \sum_{f'(i)=2} \frac{1}{u_i} + O(u_1), \quad u_1 \rightarrow 0. \end{aligned}$$

Similarly, from (2.5) we obtain

$$\begin{aligned} \left(x_2 + \frac{u_1}{t^2} + \frac{t^2}{u_1}\right) \prod_{f'(i)=1} \rho\left(\frac{u_i}{u_1}\right) = \\ \frac{t^2}{u_1} + x_2 + \frac{t(q-1)(q-t)}{q} \sum_{f'(i)=1} \frac{1}{u_i} + O(u_1), \quad u_1 \rightarrow 0. \end{aligned}$$

Therefore,

$$\begin{aligned} I_U(x_1, x_2) = \\ \left(\frac{t+t^2}{u_1} + x_1 + x_2 + \frac{t(q-1)(q-t)}{q} \sum_{i=2}^d \frac{1}{u_i}\right) I_{\{u_2, \dots, u_d\}}(x_1, x_2) + O(u_1), \end{aligned}$$

as $u_1 \rightarrow 0$, which means that both the u_1^{-1} and u_1^0 terms are symmetric in x_1 and x_2 . In other words, not only is $g(u_1)$ regular at $u_1 = 0$ but it also vanishes at $u_1 = 0$. Similarly, $g(u_1)$ is regular at ∞ and thus, as a function regular everywhere and vanishing at one point, g is zero identically. \square

3 Specialization to interpolation Macdonald polynomials

3.1 Limit transition

Let λ be the diagram of a partition of d . The diagram λ has d squares and it will be convenient to consider the set U to be indexed by the squares of λ rather than integers $\{1, \dots, d\}$. Denote by $I_\lambda(X; q, t, s, r)$ the image of $I_U(X)$ under the following specialization of the variables U

$$u_\square = -s q^{\text{col}(\square)} r^{\text{row}(\square)}, \quad \square \in \lambda. \quad (3.1)$$

Here s and r are parameters, $\square \in \lambda$ is a square in the diagram λ , and $\text{col}(\square)$ and $\text{row}(\square)$ are the column and row numbers of \square , respectively. Also, the functions f will now be functions from the diagram λ to $\{1, \dots, n\}$. Such functions are known in combinatorics as *tableaux* on λ .

Let $R_f(\lambda; q, t, r)$ be the image of R_f under the specialization (3.1). Because only the ratios of the u 's enter the formula for R_f , this specialization does not depend on s . This image is nontrivial only for some special f 's as the following proposition shows:

Proposition 1. *We have*

$$R_f(\lambda; q, t, r) = 0$$

unless f is row decreasing, that is, unless the values of f weakly decrease inside the rows.

Proof. Suppose f is not row decreasing. Then there exist two squares \square and \square_{left} in λ such that \square_{left} is just left of \square and

$$f(\square_{\text{left}}) < f(\square).$$

In this case, $R_f(\lambda; q, t, r)$ has the factor of

$$\rho\left(\frac{u_{\square_{\text{left}}}}{u_{\square}}\right) = \rho\left(\frac{1}{q}\right) = 0,$$

and hence $R_f(\lambda; q, t, r) = 0$. □

The main result of this Section is the following:

Theorem 2. *We have*

$$I_{\lambda}(X; q, t, s) \propto \lim_{r \rightarrow t^{-1}} (1 - rt)^{\ell(\lambda)-1} I_{\lambda}(X; q, t, s, r), \quad (3.2)$$

where $I_{\lambda}(X; q, t, s)$ is the interpolation Macdonald polynomial corresponding to the partition λ and the proportionality constant lies in $\mathbb{Q}(q, t)$.

Theorem 2 will be established by comparing the limit in (3.2) with the right-hand side of (1.2), which in view of the symmetry of the I_U established above, simultaneously proves (3.2) and the combinatorial formula.

First, we check that only those f which are reverse tableaux contribute to the limit (3.2). Then, in the next subsection, we shall check that for such functions f the weights

$$R_f(\lambda; q, t) = \lim_{r \rightarrow t^{-1}} (1 - rt)^{\ell(\lambda)-1} R_f(\lambda; q, t, r) \quad (3.3)$$

are proportional to the weights Ψ_f in the combinatorial formula (1.2).

Proposition 2. *We have*

$$R_f(\lambda; q, t) = 0$$

unless f is a reverse tableaux on λ . If f is a reverse tableaux then $R_f(\lambda; q, t)$ is finite and nonzero.

Proof of Proposition 2. Recall that $\rho(v)$ has two poles $v = 1, t^{-1}$ and two zeros $v = q^{-1}, qt^{-1}$. The pole $v = 1$ is not an issue in limit (3.3) and the zero $v = q^{-1}$ has been already accounted for in Proposition 1. The pole $v = t^{-1}$ means that

$$\rho\left(\frac{u_{\square}}{u_{\square_{\text{up}}}}\right) \rightarrow \infty, \quad r \rightarrow \frac{1}{t},$$

where \square_{up} is the square right on top of the square \square . This factor is present in $R_f(\lambda; q, t, r)$ if

$$f(\square) < f(\square_{\text{up}}).$$

Similarly, the zero $v = qt^{-1}$ means that

$$\rho\left(\frac{u_{\square}}{u_{\square_{\text{up}\&\text{left}}}}\right) \rightarrow 0, \quad r \rightarrow \frac{1}{t},$$

where $u_{\square_{\text{up}\&\text{left}}}$ is the square to the left of \square_{up} . This factor enters $R_f(\lambda; q, t, r)$ if

$$f(\square) < f(\square_{\text{up}\&\text{left}}).$$

Let ord_f be the order of the pole of $R_f(\lambda; q, t, r)$ at $r = t^{-1}$. We will show that

$$\text{ord}_f \leq \ell(\lambda) - 1 \quad (3.4)$$

and that this maximum is reached precisely for reverse tableaux. We have

$$\text{ord}_f = \#\{\square, f(\square) < f(\square_{\text{up}})\} - \#\{\square, f(\square) < f(\square_{\text{up}\&\text{left}})\}. \quad (3.5)$$

The contribution of any square $\square \in \lambda$ to (3.5) is 0 or ± 1 .

If the square \square is not in the first column of λ then its contribution to (3.5) is ≤ 0 . Indeed, if $f(\square) < f(\square_{\text{up}})$ then since f is row decreasing we also have

$$f(\square) < f(\square_{\text{up}}) \leq f(\square_{\text{up}\&\text{left}}).$$

That is, only the squares in the first column can make positive contribution to (3.5) which proves (3.4). It is also clear that if f is a reverse tableaux then $\text{ord}_f = \ell(\lambda) - 1$.

Now suppose that f is not a reverse tableaux. In order to prove that

$$\text{ord}_f < \ell(\lambda) - 1$$

it suffices to find one square whose contribution to ord_f is negative. By assumption, there exists a square \square such that $f(\square) \geq f(\square_{\text{up}})$. Additionally, we can assume that such a \square is minimal in the sense that $f(\square_{\text{left}}) < f(\square_{\text{up}\&\text{left}})$. Then since f is row decreasing we conclude that $f(\square) \leq f(\square_{\text{left}}) < f(\square_{\text{up}\&\text{left}})$ which means that the contribution of this square \square to (3.5) equals -1 . It follows that $\text{ord}_f < \ell(\lambda) - 1$. This concludes the proof. \square

3.2 Computation of $R_f(\lambda; q, t)$

3.2.1 The map \mathbf{L}

The weights $R_f(\lambda; q, t)$ are products of a large number of factors of the form $(1 - q^k t^l)$ where $k, l \in \mathbb{Z}$. We will treat such factors as single indivisible objects, ignoring the fact that $(1 - q^k t^l)$ is reducible if $\gcd(k, l) > 1$.

Therefore, it will be convenient to switch, following Macdonald, from the multiplicative to the additive notation as follows. Introduce a map \mathbf{L} from the free Abelian group (written multiplicatively) generated by the symbols $(1 - a)$ to the free Abelian group (written additively) generated by symbols a which is defined on generators by

$$\mathbf{L}((1 - a)) = a.$$

The map \mathbf{L} is also a very convenient way to deal with zeros and poles. If in an expression $A(z)$ depending on a parameter z

$$A(z) = \prod_i (1 - a_i(z))^{\pm 1}$$

m_+ parentheses in the numerator and m_- parenthesis in the denominator vanish at a certain point $z = z_0$ then $\mathsf{L}(A(z))$ will have a summand of $m_+ - m_-$ at $z = z_0$.

We agree that

$$\mathsf{L}^{-1}\left(\frac{x}{1-q}\right) = \prod_{i=0}^{\infty} (1 - q^i x),$$

which converges as a formal series in q or else converges in the usual analytic sense if $|q| < 1$. Normally, we will only encounter such linear combinations of the fractions $\frac{x}{1-q}$ which are, in fact, polynomials.

Taking into consideration the relation $(1-a) = -a(1-a^{-1})$ we see that it is permissible to make the following transformation

$$\sum \pm a_i \mapsto \sum \pm a_i^{-1} \quad (3.6)$$

provided $\prod (-a_i)^{\pm 1} = 1$, where the choice of signs is the same as in (3.6).

Finally, observe that

$$\mathsf{L}(\rho(x)) = x \frac{(q-1)(q-t)}{q}, \quad (3.7)$$

$$\mathsf{L}(\rho(x^{-1})) = x \frac{(q-1)(q-t)}{qt}. \quad (3.8)$$

These two formulas are related by the transformation (3.6).

3.2.2 Combinatorial formula for interpolation Macdonald polynomials

The combinatorial formula for the interpolation Macdonald polynomials [10] expresses the polynomial corresponding to the partition λ as a sum over all tableaux f on λ as follows

$$I_\mu = \sum_{\text{reverse tableaux } f} \Psi_f \prod_{\square \in \lambda} \left(x_{f(\square)} - \frac{sq^{\text{col}(\square)}}{tf(\square) + \text{row}(\square)} - \frac{tf(\square) + \text{row}(\square)}{sq^{\text{col}(\square)}} \right), \quad (3.9)$$

Here Ψ_f are a certain weight of associated to f which will be defined momentarily. Note that (3.9) differs by a change of variables from the formula established in [10].

Recall that a reverse tableaux on λ with entries in $\{1, \dots, n\}$ can be encoded as a sequence of diagrams

$$\lambda = \lambda_0 \supset \lambda_1 \supset \lambda_2 \supset \dots \supset \lambda_n = \emptyset, \quad (3.10)$$

such that each skew shape λ_k/λ_{k+1} is a horizontal strip. This encoding is simply the following:

$$\lambda_k = f^{-1}(\{k+1, \dots, n\}).$$

By definition,

$$\Psi_f = \prod_{k=0}^{n-1} \psi_{\lambda_k/\lambda_{k+1}},$$

where (see Example 2(b) in Section VI.7 of [4])

$$\mathbf{L}(\psi_{\lambda/\mu}) = \frac{t-q}{1-q} \sum_{1 \leq i \leq j \leq \ell(\mu)} (q^{\mu_i - \mu_j} - q^{\lambda_i - \mu_j} - q^{\mu_i - \lambda_{j+1}} + q^{\lambda_i - \lambda_{j+1}}) t^{j-i}.$$

One can check that this is in fact a polynomial in q and t .

3.3 The weights $R_f(\lambda; q, t)$

It is clear from their construction that the weights $R_f(\lambda; q, t)$ factor similarly to the factorization of $\psi_{\lambda/\mu}$

$$R_f(\lambda; q, t) = \prod_{k=0}^{n-1} \rho_{\lambda_k/\lambda_{k+1}},$$

where

$$\mathbf{L}(\rho_{\lambda/\mu}) = \frac{(q-1)(q-t)}{q} \sum_{\square \in \mu, \boxtimes \in \lambda/\mu} q^{\text{col}(\boxtimes) - \text{col}(\square)} t^{\text{row}(\square) - \text{row}(\boxtimes)} + \ell(\lambda) - \ell(\mu). \quad (3.11)$$

Here the number $\ell(\lambda) - \ell(\mu) \in \{0, 1\}$ is precisely the compensation for the singularity of $R_f(\lambda; q, t, r)$ as $r \rightarrow t^{-1}$.

Proposition 3. *The ratio $R_f(\lambda; q, t)/\Psi_f$ depends only on the diagram λ and not on the particular choice of a tableau f on λ .*

It is clear that this proposition is all what remains to prove in order to complete proof of Theorem 2.

Proof. Let us split the sum in (3.11) in two

$$\mathbb{L}(\rho_{\lambda/\mu}) = \Sigma_+ + \Sigma_- ,$$

where Σ_+ and Σ_- contains summands with nonnegative and negative number $\text{row}(\square) - \text{row}(\boxtimes)$, respectively.

Using the identity

$$\sum_{k=a}^b \sum_{l=c}^d q^{k-l} = -\frac{q}{(1-q)^2} (q^{a-c} + q^{b-d} - q^{a-d-1} - q^{b-c+1})$$

to perform the summation along the rows, we compute:

$$\Sigma_+ = \frac{q-t}{1-q} \sum_{i \leq j \leq \ell(\mu)} (q^{\lambda_i - \mu_j} + q^{\mu_i} - q^{\lambda_i} - q^{\mu_i - \mu_j}) t^{j-i} .$$

Similarly, we compute Σ_- using the formula (3.8) instead of (3.7) and we find that

$$\Sigma_- = \frac{q-t}{1-q} \sum_{i \leq j \leq \ell(\mu)} (q^{\mu_i - \lambda_{j+1}} + q^{-\mu_{j+1}} - q^{-\lambda_{j+1}} - q^{\mu_i - \mu_{j+1}}) t^{j-i} .$$

For any diagram η define ξ_η by

$$\mathbb{L}(\xi_\eta) = \frac{q-t}{1-q} \sum_{i \leq j \leq \ell(\eta)} (q^{\eta_i} + q^{-\eta_{j+1}} - q^{\eta_i - \eta_{j+1}}) t^{j-i} - \ell(\eta) + 1 .$$

Here the constant $1 - \ell(\eta)$ compensates for the appearances of $q^0 t^0$ in the expansion of $\frac{q-t}{1-q} q^{-\eta_{j+1}} t^{j-i}$ which occur precisely when $i = j < \ell(\eta)$. Also set by definition

$$\tau_{\lambda/\mu} = \frac{\rho_{\lambda/\mu}}{\psi_{\lambda/\mu}} \frac{\xi_\lambda}{\xi_\mu} .$$

It follows trivially from this definition that

$$\frac{R_f(\lambda; q, t)}{\Psi_f} = \frac{\xi_\emptyset}{\xi_\lambda} \prod_{k=0}^{n-1} \tau_{\lambda_k/\lambda_{k+1}} \quad (3.12)$$

It is clear that

$$\mathsf{L}(\tau_{\lambda/\mu}) = \frac{q-t}{1-q} \sum_{\ell(\mu) < j \leq \ell(\lambda)} \sum_{i \leq j} (q^{\lambda_i} + q^{-\lambda_{j+1}} - q^{\lambda_i - \lambda_{j+1}}) t^{j-i}.$$

Note that by our construction we have $\ell(\lambda) \leq \ell(\mu) + 1$. Therefore the above sum is either zero or else the only possible value of j in it is $j = \ell(\lambda)$. In the latter case we have $\lambda_{j+1} = 0$ which means that

$$\mathsf{L}(\tau_{\lambda/\mu}) = \frac{q-t}{1-q} \sum_{i < \ell(\lambda)} t^{\ell(\lambda)-i} = \frac{q-t}{1-q} \frac{t^{\ell(\lambda)} - 1}{t - 1}.$$

In other words,

$$\mathsf{L}(\tau_{\lambda/\mu}) = \begin{cases} 0, & \ell(\lambda) = \ell(\mu), \\ \frac{q-t}{1-q} \frac{t^{\ell(\lambda)} - 1}{t - 1}, & \ell(\lambda) = \ell(\mu) + 1. \end{cases}$$

It now clear that

$$\prod_{k=0}^{n-1} \tau_{\lambda_k/\lambda_{k+1}}$$

depends only on the first and last element in the chain (3.10) which precisely means that (3.12) does not depend on the particular choice of the tableau f on λ . \square

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